Collapse of a randomly forced particle: a comment

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Abstract

We refute the arguments by Anton in cond-mat/0004390, which set out to disprove the existence of a collapse transition for a randomly forced inelastic particle.

An eprint by Anton [1] has recently appeared on this archive, criticizing our proposal of a collapse transition for an inelastic, randomly-forced particle [2]. Our conclusions were that a particle forced by Gaussian white noise that rebounds from a wall with coefficient of restitution r will, with probability 1, dissipate all its energy and come to rest at the wall after an infinite number of collisions in a finite time, provided $r < r_c = e^{-\pi/\sqrt{3}}$. It has been pointed out that the transition is not present for certain discretizations of the Langevin equation [3], and it is also not clear how it manifests itself in experimentally realizable systems [4]. However, ref. [1] goes further, and argues that the transition is also absent for the ideal case of a pure white-noise force in continuous time. The purpose of this Comment is to point out substantial errors in Anton's arguments, which, we believe, invalidate his conclusions.

The existence of the collapse transition is supported by analytical calculations [5] by Burkhardt, Franklin, and Gawronski (henceforth BFG), who constructed a steady-state solution of the Fokker-Planck equation for a particle confined in a finite spatial interval. They observed that the solution is well-behaved when the coefficient of restitution r is in the range $r_c < r < 1$, where r_c is the critical value for collapse proposed in [2], whereas the rate of collisions of the particle with the boundary diverges as r approaches r_c from above. Anton's criticisms of this work, together with our responses, are as follows:

- *Burkhardt et al.'s probability density function is not normalizable*. This is incorrect. BFG state explicitly that their solution is normalizable for $r_c < r < 1$, and it is in fact the onset of non-normalizability as $r \to r_c^+$ that lead them to support our conclusions.

- *The system does not approach a steady state, since the moments of a time-dependent solution are divergent in time*. Anton bases this on two calculations. The first shows that the energy of an *elastic* particle ($r = 1$) diverges linearly in time, a result which is also stated in ref. [5]. The second is a calculation (incorrect, as we later shall argue) of spatial moments for an inelastic particle ($r < 1$) in a *semi-infinite* region, which diverge in time. Neither of these calculations are relevant to the case of an *inelastic* particle in a *finite* region, as studied by BFG. On the other hand, physical considerations show that these moments cannot diverge in that case. Firstly, the spatial moments are restricted due to the spatial confinement of the particle. Secondly, the energy is prevented from diverging by the fact that, at high speed v, energy is dissipated at a rate $\propto (1-r^2)v^3$ (collision rate $\propto v$, energy dissipation per collision $= (1-r^2)v^2$), whereas energy is being supplied at a constant rate [6].

- *The boundary conditions used by BFG lead to a contradiction*. The argument is as follows. The boundary condition for the probability density function $G(x, v)$ used by BFG, which corresponds to the stipulation that particles incident on the boundary at velocity $-v$ rebound at rv, is
$$vG(0, -v)dv = (rv)G(0, rv)d(rv). \tag{1}$$
Setting $v = 1$, and then replacing $r \to w$ leads to
$$G(0, w) = \frac{1}{w^2}G(0, -1). \tag{2}$$
However, contrary to Anton's claims, this does not imply that the flux $vG(0, v)$ has a non-integrable singularity at $v = 0$. The reason is that the solution must depend upon the coefficient of restitution—i.e., G has a parametric dependence upon r, which has been suppressed in the above notation. This means that, for a given value of r, eqn. (2) is only valid for a single value ($= r$) of w. If we include explicitly the r-dependence in $G(x, v; r)$, eqn (2) becomes
$$G\left(0, w; w\right) = \frac{1}{w^2}G\left(0, -1; w\right),$$
which does not imply that there is a non-integrable singularity in $vG(0, v, r)$ at constant r.

The final point leads Anton to propose alternative boundary conditions, which contradict eqn. (1) above. However, we have shown that there is no reason for believing eqn. (1) to be wrong, whereas Anton's alternative is not justified on physical grounds. We therefore believe that he has not constructed a valid solution for this system, and the ensuing conclusions in sections III and IV of ref. [1] are erroneous.

Ref. [1] contains a further, independent argument against the existence of collapse. Anton states that, when collisions with the boundary occur, the Langevin equation for a damped particle near an elastic wall reduces to the equation of motion for the inelastic, undamped particle. Comparing the discretized versions of these equations at the instant of collision with the boundary [7], we find

$$\begin{array}{rrcl} \text{Damped, elastic:} & v(t+\Delta t) & = & -v(t) - \Delta t \left\{ \eta_1(t) - (1-r)v(t) \right\} \\ \text{Undamped, inelastic:} & v(t+\Delta t) & = & -rv(t) - \Delta t r \eta(t) \end{array}$$

These are far from equivalent—the inelastic particle loses a finite fraction of its energy at the instant of collision, whereas the damped particle loses an infinitesimal fraction of its energy. The equations become the same if one sets $\Delta t = 1$, but this does not allow for a systematic approach to the continuum limit since the Langevin equation has already been adimensionalized (i.e., the timescale has already been set in eqn. (6) of ref. [1]). In any case, it is already known [3, 4] that inelastic collapse is destroyed by such crude discretization schemes.

To summarize, we do not believe that Anton has provided a substantive argument against the existence of the collapse transition. However, we agree that further work is needed to investigate whether, in the ideal system, inelastic collapse is definitive (i.e., is a particle that comes to rest at the wall capable of escaping again in a finite time?), and also what remnant of the collapse transition occurs in real systems.

We would like to thank Lucian Anton for interesting discussions.

References

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[5] T. W. Burkhardt, J. Franklin, and R. R. Gawronski, Phys. Rev. E **61**, 2376 (2000).

[6] The velocity performs a random walk, so the set of trajectories where v^2 increases as t^μ, with $\mu > 1$, is of measure zero.

[7] Equation (6) in [1] apparently contains a misprint, and should read $\ddot{x} = -(1-r)\dot{x} + \eta_1(t)$.